UNIT 3
Fourier Series

3.1: ORTHOGONAL FUNCTIONS

Definition: Functions $y_1(x), y_2(x), \ldots$ defined on some interval are called **orthogonal** on a≤ x ≤ b with respect to the weight function p(x) > 0 if $\int_a^b p(x) y_m(x) y_n(x) dx = 0$ for m ≠ n. $\int_a^b p(x) y_n^2(x) dx$ is called the square of the norm of $y_n(x)$ and written as $\|y_n\|^2$. If the norm is unity, we say that the set of functions is an **orthonormal** set. If the weight function is unity, we simply say that the set is **orthogonal** on a≤ x ≤ b.

Examples of sequences of orthogonal functions are: The set of functions 1, cos (nx) , n = 1, 2, 3, … or 1, sin(nx) , n = 1, 2, 3, … or 1, cos(nx), sin(nx), n = 1, 2, 3, …, on the interval $a \le x \le a + 2\pi$ with weight functions p(x) = 1 for any real constant a.

The chief advantage of the knowledge of these orthogonal sets of functions is that they yield series expansions of a given function in a simple fashion. Let $y_1, y_2, \ldots$ be an orthogonal set with respect to the weight function p(x) on an interval $a \le x \le b$. Let f(x) be a given function that can be represented in terms of $y_n(x)$ by a convergent series,

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + a_2 y_2(x) + \ldots$$

This is called an **orthogonal expansion** or the **generalized Fourier series**. The orthogonality of the functions helps us to find the unknown coefficients $a_0, a_1, a_2 \ldots$ in a simple fashion. These are called Fourier coefficients of f(x) with respect to $y_0, y_1, y_2, \ldots$. If we multiply both side of the above expansion by $p(x)y_n(x)$ for a fixed n, and then integrate over a ≤ x ≤ b, we obtain, assuming term by term integration is permissible,

$$\int_a^b p(x) f(x) y_n(x) dx = \int_a^b p(x) \left(\sum_{m=0}^{\infty} a_m y_m(x) \right) y_n(x) dx$$

$$= \sum_{m=0}^{\infty} a_m \int_a^b p(x) y_m(x) y_n(x) = a_n \int_a^b p(x) y_n^2(x) dx = a_n \|y_n(x)\|^2$$, all other

integrals being zero in the right hand side, because of the orthogonality of the set.

Thus $a_n = \dfrac{\int_a^b p(x)g(x)y_n(x)dx}{\|y_n(x)\|^2}$.

Definition: A function f is said to be a periodic function with period p if p is the least positive number such that f(x +p) = f(x) for all x in the domain of f. It follows that ***f(x+np) = f(x)*** for all x in the domain of f and all integers n.

Example : f(x) = sinx and g(x) = cos x are the familiar periodic functions with period p = 2π. The constant function h(x) = c is a periodic function, since h(x +p) =c = h(x) for all p∈ (0, ∞).

Proposition: If f and g are periodic functions with period p, then ***H(x) = af(x) + bg(x)***, for some constants a & b, is a periodic function with period p.

Proof: H(x+p) = a f(x+p) +bg(x +b) = af(x) + bg(x) = H(x) since f(x+p) = f(x) &g(x+p) = g(x)
Therefore, H(x) is a periodic function with period p.

Definition : A functional series of the form

$$\frac{a_0}{2} + a_1\cos x + b_1\sin x + a_2\cos 2x + b_2\sin 2x + \ldots + a_n\cos nx + b_n\sin nx + \ldots$$

$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ is called a **trigonometric series**; where a_0, a_n, b_n (n = 1, 2, 3, … are real constants,) are called the **coefficients** of the trigonometric series.

If the series converges, say to a function f(x), i.e, $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) = f(x)$

Then f is a period function with period p = 2π , by the above proposition.

Thus f(x+2π) = f(x) for all x in the domain of f, where $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$.

3.2 Fourier Series

3.2.1 Fourier Series of function with period 2π

Definition : The Fourier series for the periodic function f(x) in an interval α < x < α + 2π is given by.

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ And the coefficients

a_0, a_n *and* b_n, n = 1, 2, 3, … are called the **Fourier coefficients**.

To evaluate the Fourier Coefficients, the following integrals, involving sine and cosine functions are useful.

i) $\int_{\alpha}^{\alpha+2\pi}\cos nx dx = \int_{\alpha}^{\alpha+2\pi}\sin nx = 0$, n = 1, 2, 3, …

ii) $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} [\cos(m+n)x + \cos(m-n)x]dx$

$= \frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x + \frac{1}{m-n}\sin(m-n)x\right] \Big|_{\alpha}^{\alpha+2\pi} = 0$, m ≠ n.

iii) $\int_{\alpha}^{\alpha+2\pi} \cos mx \sin nx dx = 0$

iv) $\int_{\alpha}^{\alpha+2\pi} \cos^2(nx)dx = \int_{\alpha}^{\alpha+2\pi} \sin^2(nx)dx = \pi, n \neq 0$

$\int_{\alpha}^{\alpha+2\pi} \cos^2(nx)dx = \int_{\alpha}^{\alpha+2\pi} \frac{1+\cos(2nx)}{2}dx = \frac{1}{2}\left[x + \frac{1}{2n}\sin 2x\right] \Big|_{\alpha}^{\alpha+2\pi} = \frac{1}{2}[a + 2\pi - a] = \pi$

In addition to these properties of integrals involving sine and cosine functions, we often need the following trigonometric functions for particular arguments.

i) $\sin(2n+1)\frac{\pi}{2} = \cos n\pi = (-1)^n$ and (ii) $\sin n\pi = \cos(2n+1)\frac{\pi}{2} = 0$, n = 1, 2, …

Theorem : (Euler's Formulae): The Fourier coefficients in

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ are given by

$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx, a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx dx$ and $b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx dx$

Corollary : 1. If α = 0, the interval becomes 0 < x < 2π , and Euler's formulae are given by:

$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx dx$, $b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nx dx$.

2. If α = -π , then the interval becomes -π< x <π , and the Euler's Formulae and given by:

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$.

Examples If $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the range (0, 2π), show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$.

Solution: The Fourier serves for f in (0, 2π) is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, where

$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)dx = \frac{1}{\pi}\int_{\pi}^{2\pi}\left(\frac{\pi - x}{2}\right)^2 dx = \frac{1}{4\pi}\int_{0}^{2\pi}(\pi^2 - 2\pi x + x^2)dx = \frac{1}{4\pi}\left[\pi^2 x - \pi x^2 + \frac{1}{3}x^3\right]\Big|_{0}^{2\pi} = \frac{\pi^2}{6}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \cos nxdx = \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)^2 \cos nxdx)$$

$$= \frac{1}{4\pi}\left[\frac{(\pi-x)^2}{n}\sin(nx) + 2(\pi - x)\left(\frac{-\cos(nx)}{n^2}\right) + 2\frac{(-\sin(nx))}{n^3}\right]\Bigg|_0^{2\pi} = \frac{1}{4\pi}\left(\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right) = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \sin(nx)dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{-\cos(nx)}{n^2}\right) + 2(\pi - x)\left(\frac{-\sin(nx)}{n^2}\right) + \frac{2}{n^3}\cos(nx)\right]\Bigg|_0^{2\pi} = 0$$

Therefore, $f(x) = \left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}$

DIRICHLET'S CONDITIONS:

Suppose that:

a) f(x) is defined and single – valued except possibly at a finite number of points in(α, α + 2π).

b) f(x) is periodic outside (α, α + 2π) with period 2π.

c) f(x) and $f'(x)$ are sectionally continuous in (α, α + 2π).

Then the series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)$ with coefficients

$a_0 = \frac{1}{\pi}\int_a^{a+2\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\cos(nx)dx$ & $b_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\sin(nx)dx$ converges to

i) f(x) if x is a point of continuity

ii) $\dfrac{\lim\limits_{x\to c^-} f(x) + \lim\limits_{x\to c^+} f(x)}{2}$ if c is a point of discontinuity .

Moreover, when f(x) has finite number of discontinuities in any one period, for instance if in an interval (α, α + 2π), f(x) is defined by $f(x) = \begin{cases} \phi(x)\ ,\ for\ a < x < c \\ \psi(x),\ for\ c < x < a + 2\pi \end{cases}$, i.e., c is a point of discontinuity, then:

$$a_0 = \frac{1}{\pi}\left(\int_a^c \phi(x)dx + \int_c^{a+2\pi}\psi(x)dx\right)$$

$a_0 = \frac{1}{\pi}\left(\int_a^c \phi(x)\cos(nx)dx + \int_c^{a+2\pi}\psi(x)\cos(nx)dx\right)$ and $b_n = \frac{1}{\pi}\left(\int_a^c \phi(x)\sin(nx)dx + \int_c^{a+2\pi}\psi(x)\sin(nx)dx\right)$

Example: Find the Fourier series expansion for

$$f(x)=\begin{cases}-\pi, if & -\pi<x<0\\ x, if & 0<x<\pi\end{cases}$$ and deduce that $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}$.

Solution: Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)dx\right)$

Then $a_0=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\pi dx+\int_{0}^{\pi}xdx\right]=-\int_{-\pi}^{0}dx+\frac{1}{\pi}\int_{0}^{\pi}xdx=-\pi+\frac{1}{2\pi}x^2\Big|_0^{\pi}=-\pi+\frac{1}{2}\pi=-\frac{1}{2}\pi$

$$a_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\pi\cos(nx)dx+\int_{0}^{\pi}x\cos(nx)dx\right]=\int_{0}^{-\pi}\cos(nx)dx+\frac{1}{\pi}\int_{0}^{\pi}x\cos(nx)dx=\frac{\sin(nx)}{n}\Big|_0^{-\pi}+$$

$$\frac{1}{\pi}\left[\frac{x\sin(nx)}{n}\Big|_0^{\pi}-\int_0^{\pi}\frac{\sin(nx)}{n}dx\right]=\frac{\cos(nx)}{\pi n^2}\Big|_0^{\pi}=\frac{1}{\pi n^2}\left((-1)^n-1\right)$$

and $b_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\pi\sin(nx)dx+\int_{0}^{\pi}x\sin(nx)dx\right]=\int_{-\pi}^{0}-\sin(nx)dx+\frac{1}{\pi}\int_{0}^{\pi}x\sin(nx)dx$

$$=\frac{\cos(nx)}{n}\Big|_{-\pi}^{0}+\frac{1}{\pi}\left[\frac{-x\cos(nx)}{n}\Big|_0^{\pi}+\int_0^{\pi}\frac{\cos(nx)}{n}dx\right]=\frac{1-(-1)^n}{n}+\frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n}+\frac{Sin(nx)}{n^2}\Big|_0^{\pi}\right]$$

$$=\frac{1-(-1)^{n+1}}{n}+\frac{(-1)^{n+1}}{n}+0=\frac{1+2(-1)^{n+1}}{n}$$

Hence, $f(x)=\frac{\frac{-1}{2}\pi}{2}+\sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{\pi n^2}\cos(nx)+\frac{1+2(-1)^{n+1}}{n}\sin(nx)\right)$

$$=\frac{-\pi}{4}+\left(\frac{-2}{\pi}Cosx-\frac{2}{3^2\pi}Cos(3x)-\frac{2}{5^2\pi}Cos(5x)-\ldots\right)+\left(3Sinx-\frac{1}{2}Sin(2x)+Sin(3x)-\frac{1}{4}Sin(4x)+\frac{3}{5}Sin(5x)\ldots\right)$$

$$=\frac{-\pi}{4}-\frac{2}{\pi}\left(\frac{Cosx}{1^2}+\frac{Cos(3x)}{3^2}+\frac{Cos(5x)}{5^2}+\ldots\right)+\left(3\sin x-\frac{1}{2}Sin(2x)+Sin(3x)+\ldots\right).$$

By Dirichlet's Condition, we have that

$$\frac{\lim_{x\to0^-}f(x)+\lim_{x\to0^+}f(x)}{2}=\frac{-\pi}{4}-\frac{2}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)+(0+0+\ldots)$$

$$\Rightarrow\frac{-\pi+0}{2}=\frac{-\pi}{4}-\frac{2}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)$$

$$\Rightarrow\frac{-\pi}{4}=\frac{-2}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)\Rightarrow\left(\frac{-\pi}{4}\right)\left(\frac{-\pi}{2}\right)=\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots i.e; \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}.$$

3.2.2 Fourier Series of Functions with arbitrary period P = 2L

In many engineering problems, the period of the function required to be expanded is not 2π but some other interval say 2L. In order to apply the foregoing discussion to functions of period 2L, this interval must be converted to the length 2π . This involves only a proportional change in the scale.

Consider the periodic function f(x) defined on (α, α + 2L). To change the problem to period 2π put $t = \frac{\pi}{L}x$ which implies that $x = \frac{L}{\pi}t$. This gives when $x \in (a, a+2L)$. Thus the function f(x) of period 2L in (α, α + 2L) is transformed to function $f\left(\frac{L}{\pi}t\right)$ of period 2π in $\left(\frac{\alpha\pi}{L}, \frac{\alpha\pi}{L}+2\pi\right)$. Hence $f = \left(\frac{L}{\pi}t\right)$ can be expressed in Fourier series as:

$$g(t) = f\left(\frac{L}{\pi}t\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right),\ \text{where}$$

$$a_0 = \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{L}{\pi}t\right)dt,\quad a_n = \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{L}{\pi}t\right)\cos(nt)dt, \text{ and } b_n = \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{L}{\pi}t\right)\sin(nt)dt,\ \text{where}$$

$$\beta = \frac{a\pi}{L} \ \textit{and}\ x = \frac{L}{\pi}t.$$

Marking the inverse substitution $t = \frac{\pi}{L}x$ and noting that $dt = \frac{\pi}{L}dx$ in the above formula, the Fourier series Expansion of f(x) in the interval (α, α +2L) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(n\frac{\pi}{L}x\right) + b_n\sin\left(n\frac{\pi}{L}x\right)\right)\ \text{where}$$

$$a_0 = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)dx,\ a_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\left(n\frac{\pi}{L}x\right)\ dx \text{ and } b_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\left(n\frac{\pi}{L}x\right)\ dx$$

Corollary :

i. Putting α = 0 in these formulae, we get the corresponding Fourier Coefficients for the interval (0, 2L)

$$a_0 = \frac{1}{L}\int_{0}^{2L} f(x)dx,\ a_n = \frac{1}{L}\int_{0}^{2L} f(x)\cos\left(n\frac{\pi}{L}x\right)\ dx$$

$$b_n = \frac{1}{L}\int_{0}^{2L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx$$

ii. Putting a = - L in the above formulae, we get the results:

$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx$ are the Fourier coefficients in the interval (-L, L).

Example : Find the Fourier series of the periodic function f(x) of period 2, where $f(x) = \begin{cases} -1, \text{ for } -1 < x < 0 \\ 2x, \text{ for } 0 < x < 1 \end{cases}$ and show that $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}$ *and* $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4}$

Solution: 2L = 2 $\Rightarrow$ L = 1, and $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x)$

where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{1}\int_{-1}^{1} f(x)dx = \int_{-1}^{0} -dx + \int_{0}^{1} 2xdx = x\Big|_{0}^{-1} + x^2\Big|_{0}^{1} = -1+1=0$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx = -\int_{-1}^{0}\cos(n\pi x)dx + \int_{0}^{1} 2x\cos(n\pi x)dx$$

$$= \frac{\sin(n\pi x)}{n\pi}\Bigg|_{0}^{-1} + \frac{2}{n\pi}(x\sin(n\pi x))\Big|_{0}^{1} + \frac{2}{n^2\pi^2}\cos(n\pi x)\Bigg|_{0}^{1}$$

$$= \frac{2}{n^2\pi^2}\left((-1)^n - 1\right) = \frac{-2}{n^2\pi^2}(1-(-1)^n) = \frac{-2}{n^2\pi^2}\left(1+(-1)^{n+1}\right)$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)dx = \int_{-1}^{0} -\sin(n\pi x)dx + 2\int_{0}^{1}\sin(n\pi x)dx$$

$$= \frac{\cos(n\pi x)}{n\pi}\Bigg|_{-1}^{0} - \frac{2}{n\pi}\left[x\cos(n\pi x)\Big|_{0}^{1} - \int_{0}^{1}\cos(n\pi x)dx\right]$$

$$= \frac{1}{n\pi}\left(1-(-1)^n\right) - \frac{2}{n\pi}\left[(-1)^n - \frac{\sin(n\pi x)}{n\pi}\Bigg|_{0}^{2}\right] = \frac{1-(-1)^n - 2(-1)^n}{n\pi} = \frac{1-3(-1)^n}{n\pi} = \frac{1+3(-1)^{n+1}}{n\pi}$$

Thus $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(n\pi x) + b_n\sin(n\pi x))$

$$= 0 + \sum_{n=1}^{\infty}\left(\frac{-2\left(1+(-1)^{n+1}\right)}{n^2\pi^2}\cos(n\pi x) + \frac{1+3(-1)^{n+1}}{n\pi}\sin(n\pi x)\right)$$

$$= -\frac{4}{\pi^2}\cos\pi x + \frac{4}{\pi}\sin\pi x - \frac{2}{\pi}\sin 2\pi x - \frac{4}{3^2\pi^2}\cos 3\pi x + \frac{4}{3\pi}\sin 3\pi x$$

$$-\frac{2}{4\pi}\sin 4\pi x - \frac{4}{5^2\pi^2}\cos(5\pi x) + \frac{4}{5\pi}\sin(5\pi x) + \ldots$$

$$= -\frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2}+\frac{\cos 3\pi x}{3^2}+\frac{\cos 5\pi x}{5^2}+\dots\right)+\frac{2}{\pi}\left(2\sin \pi x-\frac{\sin 2\pi x}{2}+\frac{2}{3}\sin 3\pi x-\frac{\sin 4\pi x}{4}+\dots\right)$$

When x = 0, the series converges to $\frac{\lim_{x\to 0^+} f(x)+\lim_{x\to 0^-} f(x)}{2}=-\frac{1}{2}$

Therefore, $-\frac{1}{2}=\frac{-4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)+\frac{2}{\pi}(0-0+0-\dots)$

$$= \quad -\frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right) \Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}.$$

When $x=\frac{1}{2}$, $f\left(\frac{1}{2}\right)=1$, giving $1=\frac{-4}{\pi^2}(0+0+0+\dots)+\frac{2}{\pi}\left(2-\frac{0}{2}-\frac{2}{3}-\frac{0}{3}+\frac{2}{3}-\dots\right)$

$$=\frac{2}{\pi}\left(2-\frac{2}{3}+\frac{2}{5}-\frac{2}{7}+\dots\right)=\frac{4}{\pi}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right)\Rightarrow 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\frac{\pi}{4}$$

3.2.3 Fourier Series of Odd and Even Functions

Definition: A function f(x) is said to be odd iff f (-x) = -f(x)

A function f(x) is said to be even iff(-x) = f(x)

Example : The functions sin(nx) and tan (nx) are odd functions. Graph of odd function is symmetric about the origin.

Example : The functions cos(nx), x^2, sec(nx) are even functions. Graphs of even functions are symmetric about y-axis.

Proposition : If f(x) is a periodic function with period p = 2L, then

$$\int_{-L}^{L} f(x)dx=\begin{cases}2\int_0^L f(x)dx, & \text{if f is even}\\ 0, & \text{if f is odd}\end{cases}$$

Recall that a periodic function f(x) defined in (- L, L) can be represented by the Fourier series:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right),$$

where $a_0=\frac{1}{L}\int_{-L}^{L} f(x)dx,\ \ a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx\ \&\quad b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx$

When f(x) is an even function $a_0=\frac{1}{L}\int_{-L}^{L} f(x)dx=\frac{2}{L}\int_0^L f(x)dx,$

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx=\frac{2}{L}\int_0^L f(x)\cos\left(n\frac{\pi}{L}x\right)dx,\ \ \text{since } f(x)\cos\left(n\frac{\pi}{L}x\right)$$

is even, and $b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx=0,\ \ \text{since } f(x)\sin\left(n\frac{\pi}{L}x\right)$ is odd.

Therefore, the Fourier series expansion of a periodic even function f(x) contains only the cosine terms whose coefficients are $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(n\frac{\pi}{L}x\right)dx$

$$\Rightarrow f(x) = \frac{1}{L}\int_0^L f(x)dx + \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\cos\left(n\frac{\pi}{L}x\right)dx\right)\cos\left(n\frac{\pi}{L}x\right).$$

When f(x) is odd function: $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = 0,\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx = 0$, since $f(x)\cos\left(n\frac{\pi}{L}x\right)$ is odd, and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx$, since $f(x)\sin\left(n\frac{\pi}{L}x\right)$ is even.

Thus, if a period function f(x) is odd, its Fourier series expansion contains only the sine terms, whose coefficients are $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)\ dx$, so that

$$f(x) = \sum_{n=1}^{\infty}\underbrace{\left(\frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx\right)}_{Coefficients}\sin\left(n\frac{\pi}{L}x\right).$$

Example: Find a series to represent f(x) = x^2 in the interval $(-\ell, \ell)$. Deduce the values of i)

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \text{ and ii) } \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Solution: Let f(x) = x^2 in the interval $(-\ell, \ell)$ be represented by Fourier series as

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{\ell}x + b_n\sin\frac{n\pi}{\ell}x\right)$. Here $f(x) = (-x)^2 = x^2 = f(x), i.e, f(x)$ is even in $(-\ell, \ell)$.

Therefore, $a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{l}\int_0^l x^2dx = \frac{2l^2}{3}$

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx = \frac{2}{\ell}\int_0^\ell x^2\cos\left(\frac{n\pi}{\ell}x\right)dx = \frac{2}{n\pi}\left[x^2\sin\left(\frac{n\pi}{\ell}x\right)\Big|_0^\ell - \int_0^\ell 2x\sin\left(\frac{n\pi}{\ell}x\right)dx\right]$$

$$= \frac{4\ell}{n^2\pi^2}\left[x\cos\left(\frac{n\pi}{\ell}x\right)\Big|_0^\ell + \int_0^\ell\cos\left(\frac{n\pi}{\ell}x\right)dx\right] = \frac{4\ell}{n^2\pi^2}\left[\ell(-1)^n + \frac{\ell}{n\pi}\sin\left(\frac{n\pi}{\ell}x\right)\Big|_0^\ell\right] = \frac{4\ell^2(-1)^n}{n^2\pi^2} \text{ and } b_n = 0$$

Therefore, = $f(x) = x^2 = \frac{\ell^2}{3} + \frac{4\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi}{\ell}x\right) = \frac{\ell^2}{3} - \frac{4\ell^2}{\pi^2}\left[\frac{\cos\frac{\pi}{\ell}}{1^2} - \frac{\cos\frac{2\pi}{\ell}x}{2^2} + \frac{\cos\frac{3\pi}{\ell}x}{3^2} - \dots\right]$,

which is the required Fourier series. i) Putting x = 0, we get $f(0) = 0 = \frac{\ell^2}{3} - \frac{4\ell^2}{\pi^2}\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\right)$

i.e, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

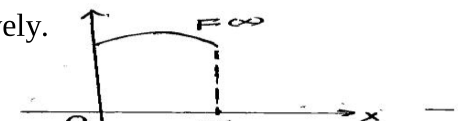
ii) Putting $x=\ell$, we get $\dfrac{\lim\limits_{x\to\ell^+} f(x)+\lim\limits_{x\to\ell^-} f(x)}{2} = \dfrac{\ell^2+\ell^2}{2=} = \ell^2 = \dfrac{\ell^2}{3} - \dfrac{4\ell^2}{\pi^2}\left(-\dfrac{1}{1^2}-\dfrac{1}{2^2}-\dfrac{1}{3^2}-\dfrac{1}{4^2}-\dots\right)$

$$\Rightarrow \underbrace{1-\frac{1}{3}}_{=\frac{2}{3}} = \frac{4}{\pi^2}\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\right) \Rightarrow \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots = \left(\frac{2}{3}\right)\left(\frac{\pi^2}{4}\right) \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

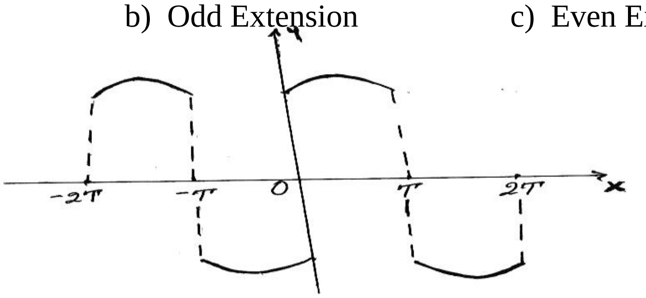
HALF – RANGE EXPANSION

In many problems of physics and engineering there is a practical need to apply a Fourier Series to a non – periodic function F(x) on the interval 0 < x < T. Because of physical or mathematical considerations, it may be possible to extend F(x) over the interval –T < x < T, making it periodic of period P = 2T. The following figures illustrate the odd and even extensions of F(x) which have Fourier Sine and Fourier Cosine series, respectively.

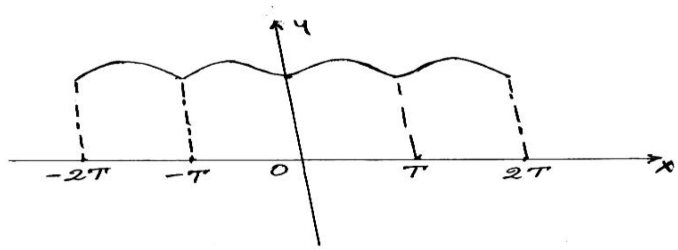
a) Original function



b) Odd Extension



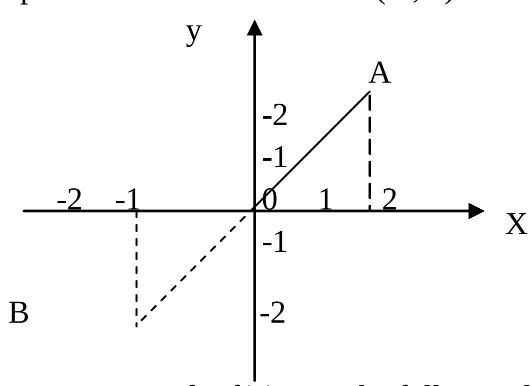
c) Even Extension



Examples : Express f(x) = x as a half – range

a) Sine series in 0 < x < 2.b)Cosine series in 0 < x < 2.

Solutions: The graph of f(x) = x in 0 < x <2 is the line OA. Let us extend the function f(x) in the interval -2 < x < 0 (shown by the line BO) so that the new function is symmetric about the origin and , therefore, represents an odd function in (-2, 2)



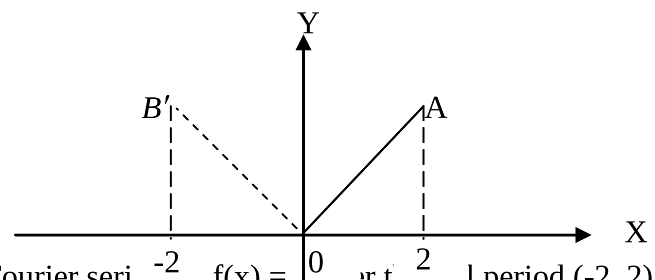
Hence the Fourier series for f(x) over the full period (-2, 2) will contain only sine series terms given by

$f(x)=\sum\limits_{n=1}^{\infty} b_n \sin\left(\dfrac{n\pi x}{2}\right)$ Where $b_n = \dfrac{2}{2}\int_0^2 f(x)\sin\left(\dfrac{n\pi x}{2}\right)dx = \int_0^2 x\sin\left(\dfrac{n\pi x}{2}\right)dx$

$$= -\left.\frac{2x\cos\left(n\frac{\pi}{2}x\right)}{n\pi}\right|_0^2 + \frac{4}{n^2\pi^2} Sin\left.\left(\frac{n\pi}{2}x\right)\right|_0^2 = \frac{-4(-1)^n}{n\pi} + 0 = \frac{4(-1)^{n+1}}{n\pi}$$

$$\Rightarrow f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{2}\right) = \frac{4}{\pi}\left[\sin\left(\frac{\pi}{2}x\right) - \frac{\sin\pi x}{2} + \frac{\sin\frac{3}{2}\pi x}{3} - \cdots\right]$$

b) The graph of f(x) = x in (0, 2) is the line OA. Let us extend the function f(x) in the interval (-2, 0) (shown by the $B'O$) so that the new function is symmetric about the y – axis and, therefore, represents an even function in (-2, 2).



Hence the Fourier seri f(x) = er t l period (-2, 2) will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{2}\right) \quad a_0 = \frac{2}{2}\int_0^2 f(x)dx = \int_0^2 xdx = 2 \text{ and } a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx = \frac{2x\sin}{n\pi}\left.\left(\frac{n\pi x}{2}\right)\right|_0^2 - \frac{2}{n\pi}\int_0^2\sin\left(\frac{n\pi x}{2}\right)dx = \frac{4}{n^2\pi^2}\cos\left.\left(\frac{n\pi x}{2}\right)\right|_0^2 = \frac{4}{n^2\pi^2}\left((-1)^2 - 1\right) = \frac{4\left((-1)^n - 1\right)}{n^2\pi^2}$$

Therefore, the desired result is: $f(x) = \frac{2}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\left(n\frac{\pi}{2}x\right)$

$$= 1 + \frac{4}{\pi^2}\left(\frac{-2}{1^2}Cos\frac{\pi}{2}x - \frac{2}{3^2}Cos\frac{3\pi}{2}x - \frac{2}{5^2}Cos\frac{5\pi}{2} + - \ldots\right) = 1 - \frac{8}{\pi^2}\left(\frac{\cos\frac{\pi}{2}}{1^2} + \frac{\cos\frac{3}{2}\pi}{3^2} + \frac{\cos\frac{5}{2}\pi x}{5^2} + \ldots\right)$$

3.3 FOURIER INTEGRALS:

Consider a function f(x) which satisfies the Dirichlets conditions

in every interval (-L, L) so that, we have $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$, where

$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt, a_n = \frac{1}{\mathrm{L}}\int_{\mathrm{-L}}^{\mathrm{L}}\mathrm{f(t)cos}\frac{\mathrm{n}\pi\mathrm{t}}{\mathrm{L}}dt$, and $b_n = \frac{1}{L}\int_{-L}^{L} f(t)sin\frac{n\pi t}{L}dt$. Substituting the values of a_0 $\mathrm{a_n}$ and $\mathrm{b_n}$ in the Fourier series expansion, we get the form

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)dt + \frac{1}{L}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt\cos\frac{n\pi x}{L} + \int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt\sin\frac{n\pi x}{L}\right)$$

$$=\frac{1}{2L}\int_{-L}^{L} f(t)dt+\frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}(t-x)\,dt \qquad \text{-- (*)}$$

If $\int_{-\infty}^{\infty}|f(t)|dx$ converges, i.e.; f(x) is absolutely integrable in (- ∞,∞), then the 1st – term on the right side of (*) approaches 0 as L→∞, since $\left|\frac{1}{2L}\int_{-L}^{L} f(t)dt\right| \le \frac{1}{2L}\int_{-\infty}^{\infty}|f(t)|dt$.

The 2nd – term on the right side of (*) tends to $\lim_{L\to\infty}\frac{1}{L}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\frac{n\pi(t-x)}{L}dt$

$=\lim_{\delta\lambda\to 0}\frac{1}{\pi}\sum_{n=1}^{\infty}\delta\lambda\int_{-\infty}^{\infty} f(t)\cos n\delta\lambda(t-x)dt$ on writing $\frac{\pi}{L}=\delta\lambda$ .Thus as L →∞ , (*) becomes

$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dtd\lambda$ called the Fourier Integral of f(x).

Remark: 1. If function f is continuous at x, then $f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dtd\lambda$.

If f is not continuous at x, then $\frac{f(x+0)+f(x-0)}{2}=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,\mathrm{dt}\mathrm{d}\lambda$

where $f(x+0)=\lim_{t\to x^{+}} f(t)\,and\ \mathrm{f(x-0)}=\lim_{\mathrm{t}\to \mathrm{x}^{-}} f(t)$.

2. Fourier sine and cosine integrals. Expanding cosλ(t-x) = cos (λt - λx) = cosλt cosλx + sin λt sin λt, the Fourier integral of f(x) may be written as

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\cos\lambda x\int_{-\infty}^{\infty} f(t)\cos\lambda t\,dt\,d\lambda+\frac{1}{\pi}\int_{0}^{\infty}\sin\lambda x\int_{-\infty}^{\infty} f(t)\sin\lambda t\,dt\,d\lambda .$$

If f(x) is an odd function, f(t) cos (λt) is also an odd function while f(t) sin (λt) is even. Then the 1st term on the right side of the above equation vanishes, and $f(x)=\frac{2}{\pi}\int_{0}^{\infty}\sin\lambda\mathrm{x}\int_{0}^{\infty}\mathrm{f(t)}\sin(\lambda\mathrm{t})\mathrm{dt}\,\mathrm{d}\lambda$, which is known as the Fourier Sine integral.

Similarly, if f(x) is even, the above integral takes the form

$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\cos(\lambda x)\int_{0}^{\infty} f(t)\cos(\lambda t)dt\,d\lambda$, known as the Fourier Cosine integral.

Example: Express $f(x)=\begin{cases}1,\ \text{for } 0\le \mathrm{x}\le\pi\\ 0,\ \text{for x} > \pi\end{cases}$ as a Fourier sine integral and hence

evaluate $\int_{0}^{\infty}\frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)d\lambda$.

Solution: The Fourier sine integral of f(x) is $f(x)=\frac{2}{\pi}\int_0^{\infty}\sin(\lambda x)\,d\lambda\int_0^{\infty}\mathrm{f(t)sin(\lambda t)dt}$

$$=\frac{2}{\pi}\int_0^{\infty}\sin(\lambda x)\,d\lambda\int_0^{\pi}\mathrm{sin(\lambda t)dt}=\frac{2}{\pi}\int_0^{\infty}\sin(\lambda x)d\lambda\left[\left.\frac{\cos(\lambda t)}{\lambda}\right|_{\pi}^{0}\right]=\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos(\lambda\pi)}{\lambda}\sin(x\lambda)d\lambda$$

Therefore, $f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos(\lambda\pi)}{\lambda}\sin(\lambda x)\,d\lambda\Rightarrow\int_0^{\infty}\frac{1-\cos(\lambda x)}{\lambda}\sin(\lambda x)\,d\lambda=\frac{\pi}{2}f(x)=\begin{cases}\frac{\pi}{2}, & \text{for } 0\le \mathrm{x}\le\pi\\ 0, & \text{for x}>\pi\end{cases}$

At x =π, which is a point of discontinuity of f(x), then the value of the above integral is

$$\frac{\pi}{2}\left(\frac{\lim\limits_{x\to\pi^+}f(x)+\lim\limits_{x\to\pi^-}f(x)}{2}\right)=\frac{\pi}{2}\left(0+\frac{1}{2}\right)=\frac{\pi}{4}, \text{ i.e, } \int_0^{\infty}\frac{1-\cos(\lambda\pi)}{\lambda}\sin\left(\lambda\mathrm{x}\right)d\lambda=\frac{\pi}{4}, at\ \mathrm{x}=\pi$$